

Gaussian fluctuations for stochastic Volterra equations with small noise

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Content

This report is based on joint works with N. T. Dung.

The content of the report consists of 3 parts:

1. Introduction
2. Main results
3. Example

1. Introduction

For every $\varepsilon \in (0, 1)$, we consider stochastic Volterra equations of the form

$$X_{\varepsilon,t} = x_0 + \int_0^t b(t, s, X_{\varepsilon,s}) ds + \varepsilon \int_0^t \sigma(t, s, X_{\varepsilon,s}) dB_s, \quad 0 \leq t \leq T, \quad (1)$$

where the initial data $x_0 \in \mathbb{R}$, $(B_t)_{t \in [0, T]}$ is a standard Brownian motion and b, σ are measurable functions on $[0, T]^2 \times \mathbb{R}$.

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- This class of stochastic Volterra equations has been widely used in the modelling of numerous random phenomena (fluid turbulence [4], DNA patterns [14] and mathematical finance [11], etc)
- The asymptotic behavior of $X_{\varepsilon,t}$ as $\varepsilon \rightarrow 0$ has been investigated by various authors. However, most of papers are devoted to the large and moderate deviation results, see e.g. [9, 10, 13, 16]

1. Introduction

We observe that, as ε tends to 0, the solution $(X_{\varepsilon,t})_{t \in [0,T]}$ of the equation (1) converges to a deterministic process $(x_t)_{t \in [0,T]}$ solving the following integral equation

$$x_t = x_0 + \int_0^t b(t, s, x_s) ds, \quad 0 \leq t \leq T. \quad (2)$$

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$$\tilde{X}_{\varepsilon,t} := \frac{X_{\varepsilon,t} - x_t}{\varepsilon}, \quad 0 \leq t \leq T. \quad (3)$$

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It is known from [15] that $\tilde{X}_{\varepsilon,t}$ converges to a Gaussian limit Y_t given by

$$Y_t = \int_0^t b'(t, s, x_s) Y_s ds + \int_0^t \sigma(t, s, x_s) dB_s, \quad 0 \leq t \leq T. \quad (4)$$

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$$d_{TV}(\tilde{X}_{\varepsilon,t}, Y_t) := \sup_A |P(\tilde{X}_{\varepsilon,t} \in A) - P(Y_t \in A)| = \frac{1}{2} \sup_g |Eg(\tilde{X}_{\varepsilon,t}) - Eg(Y_t)|,$$

where the supremum is running over all Borel subsets A of the real line and all measurable functions bounded by 1.

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where the supremum is running over all Borel subsets A of the real line and all measurable functions bounded by 1.

Then, we use the techniques of Malliavin calculus to obtain an explicit estimate for $d_{TV}(\tilde{X}_{\varepsilon,t}, Y_t)$. Furthermore, we also show that the rate of convergence is optimal order.

1. Introduction

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Lemma 1.1 (Theorem 3.1 in [5]).

Let $F_1 \in \mathbb{D}^{2,4}$ be such that $\|DF_1\|_{L^2[0,T]} > 0$ a.s. Then, for any random variable $F_2 \in \mathbb{D}^{1,2}$ and any measurable function g with $\|g\|_\infty = \sup_{x \in \mathbb{R}} |g(x)| \leq 1$, we have

$$|Eg(F_1) - Eg(F_2)| \leq C \left(E \|DF_1\|_{L^2[0,T]}^{-8} E \left(\int_0^T \int_0^T |D_\theta D_r F_1|^2 d\theta dr \right)^2 + (E \|DF_1\|_{L^2[0,T]}^{-2})^2 \right)^{\frac{1}{4}} \|F_1 - F_2\|_{1,2}, \quad (5)$$

provided that the expectations exist, where C is an absolute constant.

We consider the random variables $F_1 = Y_t$ and $F_2 = \tilde{X}_{\varepsilon,t}$.

2. The main results

Assumption A1.

The coefficients $b, \sigma : [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ have linear growth, i.e.

$$|b(t, s, x)| + |\sigma(t, s, x)| \leq k_1(t, s)(1 + |x|) \quad \forall x \in \mathbb{R}, 0 \leq s, t \leq T,$$

and $b(t, s, x), \sigma(t, s, x)$ are differentiable functions in x with bounded derivatives

$$|b'(t, s, x)| + |\sigma'(t, s, x)| \leq k_2(t, s) \quad \forall x, y \in \mathbb{R}, 0 \leq s, t \leq T,$$

where $k_1(t, s)$ and $k_2(t, s)$ are non-negative measurable functions on $[0, T] \times [0, T]$ such that

$$\sup_{t \in [0, T]} \int_0^t \left(k_1^{2\alpha}(t, s) + k_2^{2\beta}(t, s) \right) ds \leq L < \infty \quad (6)$$

for some $L > 0$ and $\alpha, \beta > 1$.

2. The main results

Assumption A2.

$b(t, s, x), \sigma(t, s, x)$ are twice differentiable functions in x , and satisfy

$$|b''(t, s, x)| + |\sigma''(t, s, x)| \leq k_3(t, s) \quad \forall x \in \mathbb{R}, 0 \leq s, t \leq T,$$

where $k_3(t, s)$ is non-negative measurable function on $[0, T] \times [0, T]$ such that

$$\sup_{t \in [0, T]} \int_0^t k_3^{2\gamma}(t, s) ds \leq L < \infty \quad (7)$$

for some $L > 0$ and $\gamma > 1$.

2. The main results

Proposition 2.1.

Let Assumption (A_1) hold. Let $(X_{\varepsilon,t})_{t \in [0,T]}$ be the solution to the equation (1). Then, for each $0 \leq t \leq T$, the random variable $X_{\varepsilon,t}$ is Malliavin differentiable. Moreover, the derivative $D_\theta X_{\varepsilon,t}$ satisfies $D_\theta X_{\varepsilon,t} = 0$ for $\theta > t$ and

$$D_\theta X_{\varepsilon,t} = \varepsilon \sigma(t, \theta, X_{\varepsilon,\theta}) + \int_\theta^t b'(t, s, X_{\varepsilon,s}) D_\theta X_{\varepsilon,s} ds \\ + \varepsilon \int_\theta^t \sigma'(t, s, X_{\varepsilon,s}) D_\theta X_{\varepsilon,s} dB_s, \quad 0 \leq \theta \leq t \leq T. \quad (8)$$

2. The main results

Proposition 2.2.

Suppose Assumption (A_1) . Let $(X_{\varepsilon,t})_{t \in [0,T]}$ be the solution to the equation (1). Then, we have

$$\int_0^t E |D_\theta X_{\varepsilon,t}|^2 d\theta \leq C \left(\sup_{0 \leq u \leq t} \int_0^u k_1^2(u, \theta) d\theta \right) \varepsilon^2 \quad \forall \varepsilon \in (0, 1), \quad (9)$$

where C is a positive constant not depending on t and ε .

2. The main results

For each $t \in [0, T]$, the random variable Y_t is Malliavin differentiable and its derivative is given by $D_\theta Y_t = 0$ for $\theta > t$ and

$$D_\theta Y_t = \sigma(t, \theta, x_\theta) + \int_\theta^t b'(t, s, x_s) D_\theta Y_s ds, \quad 0 \leq \theta \leq t \leq T. \quad (10)$$

$D_\theta Y_t$ are deterministic so $D_r D_\theta Y_t = 0$, $0 \leq r, \theta \leq t \leq T$.

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$D_\theta Y_t$ are deterministic so $D_r D_\theta Y_t = 0$, $0 \leq r, \theta \leq t \leq T$. Moreover,

$$\|DF_1\|_{L^2[0, T]}^2 = \|DY_t\|_{L^2[0, T]}^2 = \text{Var}(Y_t).$$

We have

$$\|\tilde{X}_{\varepsilon, t} - Y_t\|_{1,2} = \left(E|\tilde{X}_{\varepsilon, t} - Y_t|^2 + E\|D\tilde{X}_{\varepsilon, t} - DY_t\|_{L^2[0, T]}^2 \right)^{\frac{1}{2}}$$

2. The main results

Proposition 2.3.

Suppose Assumption (A_1) and Assumption (A_2) . Consider the stochastic processes $(\tilde{X}_{\varepsilon,t})_{0 \leq t \leq T}$ and $(Y_t)_{0 \leq t \leq T}$ defined by (3) and (4), respectively. For every $p \geq 2$, we have

$$E|\tilde{X}_{\varepsilon,t} - Y_t|^p \leq C \left(\left(\sup_{0 \leq u \leq t} \int_0^u k_2^2(u,s) ds \right)^{\frac{p}{2}} + \left(\sup_{0 \leq u \leq t} \int_0^u k_3^2(u,s) ds \right)^{\frac{p}{2}} \right) \\ \times \left(\sup_{0 \leq u \leq t} \int_0^u k_1^2(u,s) ds \right)^{\frac{p}{2}} \varepsilon^p \quad \forall \varepsilon \in (0,1), 0 \leq t \leq T, \quad (11)$$

where C is a positive constant not depending on t and ε .

2. The main results

Proposition 2.4.

Suppose Assumptions (A_1) and Assumption (A_2) . Consider the stochastic processes $(\tilde{X}_{\varepsilon,t})_{0 \leq t \leq T}$ and $(Y_t)_{0 \leq t \leq T}$ defined by (3) and (4). Then we have

$$\begin{aligned} E \|D\tilde{X}_{\varepsilon,t} - DY_t\|_{L^2[0,T]}^2 &\leq C \left(\left(\sup_{0 \leq u \leq t} \int_0^u k_2^2(u,s) ds \right) + \left(\sup_{0 \leq u \leq t} \int_0^u k_3^2(u,s) ds \right) \right) \\ &\quad \times \left(\sup_{0 \leq u \leq t} \int_0^u k_1^2(u,s) ds \right) \varepsilon^2 \quad \forall \varepsilon \in (0,1), 0 \leq t \leq T, \quad (12) \end{aligned}$$

where C is a positive constant not depending on t and ε .

2. The main results

We obtain the estimate in the following Theorem:

Theorem 2.5.

Suppose Assumptions (A_1) and (A_2) . We consider the stochastic processes $(\tilde{X}_{\varepsilon,t})_{0 \leq t \leq T}$ and $(Y_t)_{0 \leq t \leq T}$ defined by (3) and (4), respectively. Then, for all $\varepsilon \in (0, 1)$, $0 < t \leq T$, we have

$$d_{TV}(\tilde{X}_{\varepsilon,t}, Y_t) \leq \left(\left(\sup_{0 \leq u \leq t} \int_0^u k_2^2(u, s) ds \right) + \left(\sup_{0 \leq u \leq t} \int_0^u k_3^2(u, s) ds \right) \right)^{\frac{1}{2}} \\ \times \left(\sup_{0 \leq u \leq t} \int_0^u k_1^2(u, s) ds \right)^{\frac{1}{2}} \frac{C\varepsilon}{\sqrt{\text{Var}(Y_t)}},$$

where C is a positive constant not depending on t and ε .

2. The main results

Theorem 2.6.

Suppose Assumptions (A_1) and (A_2) . We additionally assume that $b''(t, s, x)$ are continuous in x . Then, for any continuous and bounded function φ , we have

$$\lim_{\varepsilon \rightarrow 0} \frac{E\varphi(\tilde{X}_{\varepsilon,t}) - E\varphi(Y_t)}{\varepsilon} = \frac{1}{2\text{Var}(Y_t)} E[\varphi(Y_t)\delta(Z_t DY_t)], \quad 0 < t \leq T, \quad (13)$$

where the stochastic process $(Z_t)_{0 \leq t \leq T}$ is given by

$$Z_t = \int_0^t b'(t, s, x_s) Z_s ds + \int_0^t b''(t, s, x_s) Y_s^2 ds + 2 \int_0^t \sigma'(t, s, x_s) Y_s dB_s, \quad t \in [0, T]. \quad (14)$$

3. Proof of Theorem 2.2

We will carry out the proof in two steps.

Step 1. we show that for all $p \geq 2$ then

$$\lim_{\varepsilon \rightarrow 0} E \left| \frac{\tilde{X}_{\varepsilon,t} - Y_t}{\varepsilon} - \frac{1}{2} Z_t \right|^p = 0, \quad 0 \leq t \leq T. \quad (15)$$

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Step 2. In this step, we prove (13).

$$\lim_{\varepsilon \rightarrow 0} \frac{E\varphi(\tilde{X}_{\varepsilon,t}) - E\varphi(Y_t)}{\varepsilon} = \frac{1}{2\text{Var}(Y_t)} E[\varphi(Y_t)\delta(Z_t DY_t)], \quad 0 < t \leq T.$$

3. Example

We consider the following stochastic Volterra equation:

$$X_{\varepsilon,t} = x_0 + \int_0^t K_H(t,s)b(s, X_{\varepsilon,s})ds + \varepsilon \int_0^t K_H(t,s)\sigma(s, X_{\varepsilon,s})dB_s, \quad (16)$$

where $\varepsilon \in (0, 1)$, the initial data $x_0 \in \mathbb{R}$ and

$$K_H(t,s) := \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H-1/2)\sqrt{V_H}} F(H-1/2, 1/2-H, H+1/2, 1-t/s), \quad 0 \leq s < t \leq T, \quad (17)$$

where F is the Gauss hypergeometric function, Γ denotes the usual Gamma function and $V_H = \frac{\Gamma(2-2H)\cos\pi H}{\pi H(1-2H)}$.

3. Example

We recall that the kernel K_H can be used to define a fractional Brownian motion as follows (see e.g. [3])

$$B_t^H = \int_0^t K_H(t, s) dB_s, \quad 0 \leq t \leq T. \quad (18)$$

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When $H \geq \frac{1}{2}$, the kernel K_H reduces to the following (see e.g. [12] pp. 277-279)

$$K_H(t, s) := c_H s^{1/2-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-1/2} du, \quad 0 \leq s < t \leq T, \quad (19)$$

where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-1/2)}}$ with β is the Beta function.

3. Example

We need the assumptions that $b, \sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions having linear growth, that is, there exists $M > 0$ such that

$$|b(t, x)| + |\sigma(t, x)| \leq M(1 + |x|) \quad \forall x \in \mathbb{R}, 0 \leq t \leq T.$$

Additionally, b and σ are twice differentiable functions in x with the partial derivatives bounded by M .

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Additionally, b and σ are twice differentiable functions in x with the partial derivatives bounded by M .

We define

$$x_t = x_0 + \int_0^t K_H(t, s) b(s, x_s) ds, \quad 0 \leq t \leq T. \quad (20)$$

$$\tilde{X}_{\varepsilon, t} := \frac{X_{\varepsilon, t} - x_t}{\varepsilon}, \quad 0 \leq t \leq T, \quad (21)$$

and

$$Y_t = \int_0^t K_H(t, s) b'(s, x_s) Y_s ds + \int_0^t K_H(t, s) \sigma(s, x_s) dB_s, \quad 0 \leq t \leq T. \quad (22)$$

3. Example

We recall from [7] that $0 \leq K_H(t, s) \leq c(t-s)^{H-\frac{1}{2}}s^{-|H-\frac{1}{2}|}$ for some $c > 0$. Hence, it is easy to verify that the coefficients of the equation (16) satisfy Assumptions (A_1) and (A_2) with $k_1(t, s) = k_2(t, s) = k_3(t, s) = MK_H(t, s)$.

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Theorem .

For every $H \in (0, 1)$, it holds that

$$d_{TV}(\tilde{X}_{\varepsilon, t}, Y_t) \leq \frac{Ct^{2H}\varepsilon}{\sqrt{\text{Var}(Y_t)}}, \quad \forall \varepsilon \in (0, 1), \quad 0 < t \leq T, \quad (23)$$

where C is a positive constant not depending on t and ε . When $H \geq \frac{1}{2}$ and $\sigma(t, x) \geq \sigma_0$ for some $\sigma_0 > 0$ and for all $(t, x) \in [0, T] \times \mathbb{R}$, we also have

$$d_{TV}(\tilde{X}_{\varepsilon, t}, Y_t) \leq Ct^H\varepsilon, \quad \forall \varepsilon \in (0, 1), \quad 0 < t \leq T. \quad (24)$$

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Thank you for listening!